Diagrammatic Notations in Mathematical Proofs

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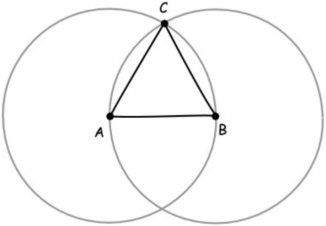
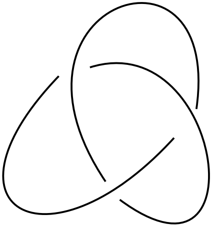
Princeton University

BIG PROOF WORKSHOP

May 29, 2019
ICMS, Edinburgh

Diagrams in Mathematics





$$\begin{tikzcd}
 & \cdots \arrow[d, "\delta_{p+2}"] & \cdots \arrow[d, "\delta_{p+2}"] & \cdots \arrow[d, "\delta_{p+2}"] & \\
0 \arrow[r] & A_{p+1} \arrow[r, "i_{p+1}"] \arrow[d, "\delta_{p+1}"] & B_{p+1} \arrow[r, "j_{p+1}"] \arrow[d, "\delta_{p+1}"] & C_{p+1} \arrow[r] \arrow[d, "\delta_{p+1}"] & 0 \\
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Aims of the Talk

1. Provide a working definition of mathematical diagrams $\rightarrow$ argue that they can enter into the inferential structure of proofs.
    a. Some diagrams enable an acceptable use of spatio-temporal intuition.
    b. Others do not involve intuition at all.

2. Suggest that the effectiveness of mathematical notations depends on the possibility of supporting specific operations $\rightarrow$ in some cases changing the notation would lead to changing the proof at issue.

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Traditional Proofs and Formal Proofs

proofs are written in a way to make them easily understood by mathematicians. Routine logical steps are omitted. An enormous amount of context is assumed on the part of the reader. (Hales, 2008)

1. Proofs are targeted to a specific audience.
   - Checking their correctness is not an easy or automatic task and mathematicians' ability to tell whether an argument is a proof is not infallible.
2. The *criteria of acceptability* for proofs are context-dependent.
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Criteria of Acceptability

The way in which we check the validity of traditional proofs is more variegated compared to checking for formal correctness and cannot be completely spelled out without entering into the details of the various cases.

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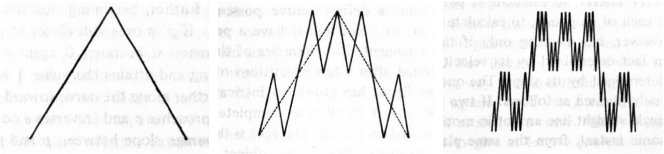
Crisis in Intuition



had we relied on intuition in this instance, we should have remained in error, for intuition seems to force the conclusion that there cannot be curves lacking a tangent at any point. (Hahn, 1933)

Topology

Proofs, especially in topology and geometry, rely on intuitive arguments in situations where a trained mathematician would be capable of translating those intuitive arguments into a more rigorous argument. (Hales, 2008)

1. Spatio-temporal intuition seems therefore to be acceptable, but only in specific situations.
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Ban on Diagrams

The ban on diagrams which came into place after the crisis in intuition is partially unwarranted:

1. Some diagrams enable acceptable uses of intuition:
   - Knot diagrams
   - Topological pictures
   - Venn diagrams
   - The graphical language for monoidal categories

2. Some diagram do not trigger intuition at all!
   - Commutative diagrams in homological algebra
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Need for a Definition

Diagrams play a central role in different areas of contemporary mathematics. Still, there are some problems:

- In the literature, the term ‘diagram’ is used in many different ways.
- Diagrams are sometimes seen just as heuristic aids, not having any justificatory role.
- When they are seen as playing a genuine role in proofs, then they are often disregarded as ‘notational variance.’
- It is sometimes argued that when diagrams can be used systematically and rigorously, they are not diagrams anymore!

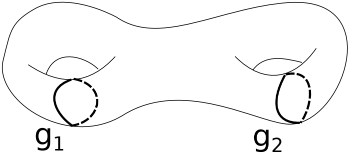
Outline

Definition of Diagrams

Definition

A *mathematical diagram* is a two-dimensional interpreted display which is an element of a mathematical notation. As with elements of notations in general, a mathematical diagram is deployed in a mathematical practice, which supplies constraints on its interpretation and on its operative dimension.

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\phi"'] & H \\ G/N \arrow[ur, "f_*"'] & \end{tikzcd}$$



What a Diagrams is Not

1. Linear displays such as linear algebraic notation and written natural language, just in virtue of their two-dimensionality:

$$x + y = z$$

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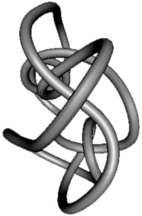
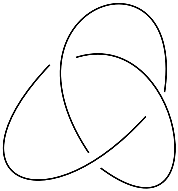


2-dimensionality

- Diagrams exploit in a non-trivial way the space of the page, the "information is indexed by location in a plane" (Larkin and Simon, 1987).
- Their planar nature does not conflict with the fact that diagrams can be coded as linear displays (e.g. in LaTeX).
- Thanks to their 2-dimensional layout, diagrams can externalize sophisticated mathematical relations. Moreover, they support multiple readings.



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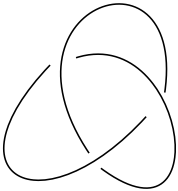

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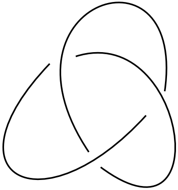

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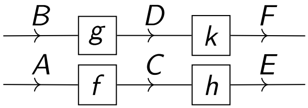
Multiple Readibility: The Graphical Language for Monoidal Categories

The tensor product $\otimes$ and composition $\circ$ are compatible.
Let A, B, C, D, E, F be objects and $f : A \to C$, $g : B \to D$, $h : C \to E$, $k : D \to F$ morphisms in a monoidal category.

$$(h \otimes k) \circ (f \otimes g) = (h \circ f) \otimes (k \circ g)$$



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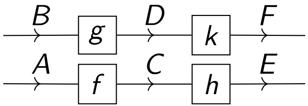

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2-dimensionality

These notations have 2-dimensional components, but they can be interpreted as linear insofar as there is a standard reading direction:

$$x^{2^{2^{2^{\cdots}}}}$$

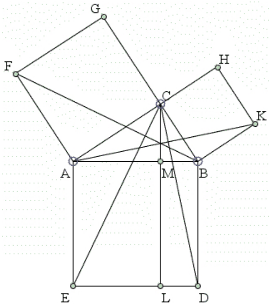
$$[a_1, \ldots, a_n, \ldots] := a_1 + \cfrac{1}{a_2 + \ldots \frac{1}{a_n + \ldots}}$$

$$\int_a^b x^2 dx$$

Mathematical Practices

Diagrams are elements of notations:

- Each notation presents clearly identifiable and reproducible *constitutive* perceptual features, e.g. straight lines, interrupted lines, arrows, dotted lines, etc. which can carry mathematical content.
- Other perceptual features are just *enabling*.



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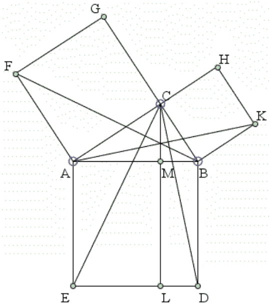

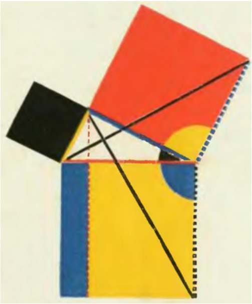

Operative Dimension

Definition

The *operative dimension* of a notation is constituted by those manipulations corresponding to mathematical operations.

In order to be at all admissible, these manipulation must be sharable and reproducible in the relevant context. Moreover, they must depend only on the constitutive features of the notation.

1. Knot diagrams: Reidemeister moves, three-dimensional moves, Kirby moves, etc.
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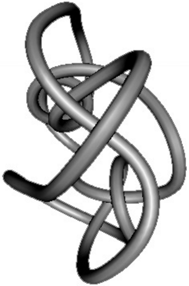
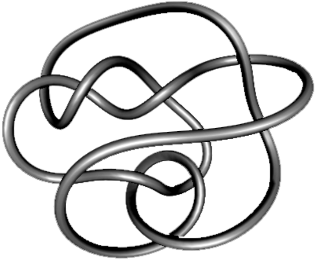
From **Illustrations** to Diagrams



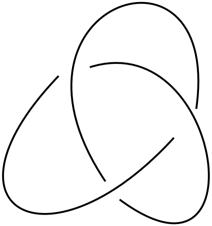


From Illustrations to **Diagrams**

Diagrams are more controlled representations in which the constitutive features are clearly identifiable.

In order to obtain a *knot diagram* one has to:

- *Project* the knot on a surface, keeping the information at crossings (singular points).
- Make sure that the projection is *regular*: the intersection points are transversal and involve two strands at a time.



A Toy Example: Untying the Knot

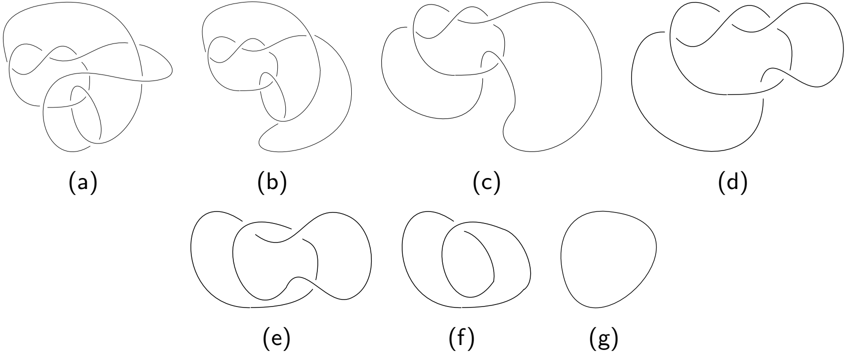


Figure: Untying the knot

Manipulative Imagination

The previous diagrams prove that the complicated initial diagram represents the unknot.

Knot diagrams trigger a special kind of imagination that does not involve only vision, but also spatial-motoric intuition of 3d space (De Toffoli and Gardino, 2014).

More than simple vision:

1. Exploits spontaneous abilities, but gets enhanced by training.
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Reliability

These manipulations performed on knot diagrams correspond to a sequence of Reidemeister moves, but they are reliably identified as not altering the knot type directly as 3-dimensional moves.

- Thanks to the convenient convention to represent the singular points, it is easy to visualize a curve in space from a knot diagram.
- The moves are sufficiently simple to be grasped by an average practitioner, but they can be decomposed in smaller ones if required.
- Imagining 3-dimensional transformations is epistemically relevant: If we imagine incorrectly we get a wrong result.
- Although intuition is linked by training to precise mathematical operations, it is not therefore dispensable.
- Arguments relying on diagrams and intuition are cognitively manageable and reliable ways to proxy more formal arguments.

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Perko Pair

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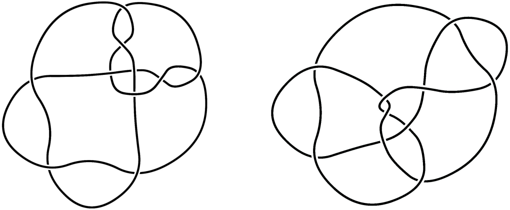

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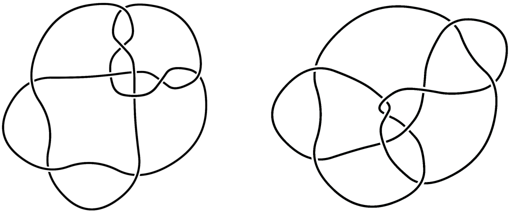

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Knot Codes

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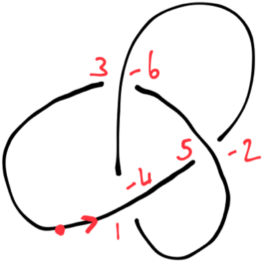

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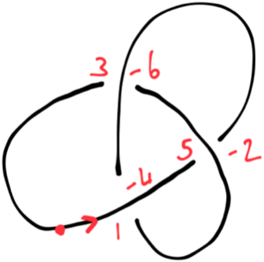

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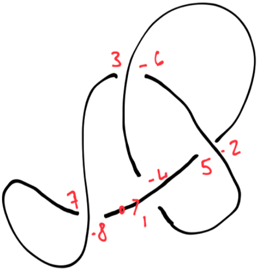
Moreover, we can recover certain moves from the code:



$$(1, -4),\ (3, -6),\ (5, -2),\ (7, -8)$$

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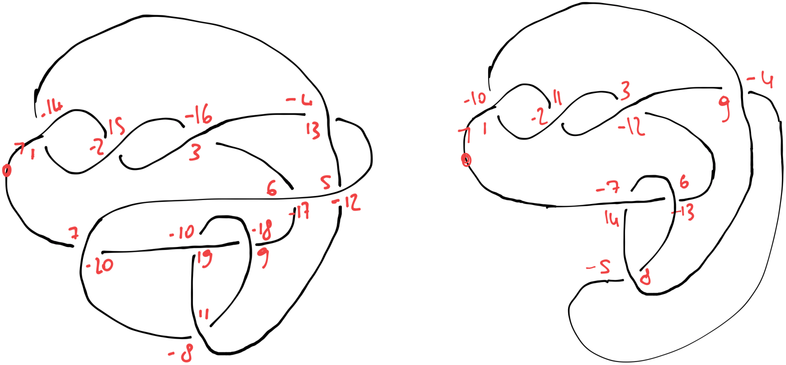
Knot Codes

First step of the proof above:



$$-14, -16, -12, -20, -18, -8, -4, -2, 6, -10$$

$$-10, -12, 8, 14, -4, -2, 6$$

Outline

"Chasing" the Diagram

Lemma (Strong version of the Five Lemma)

Let the following diagram be commutative and such that its rows are exact. If f_2 and f_4 are surjective and f_5 is injective, then f_3 is surjective. Symmetrically, if f_2 and f_4 are injective, and f_1 is surjective, then f_3 is injective.

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5
\end{tikzcd}$$

We will prove the first statement of the lemma: If f_2 and f_4 are surjective and f_5 is injective, then f_3 is surjective.

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5 \\
& & b_3 & &
\end{tikzcd}$$

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$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5 \\
 & & b_3 \arrow[r, dotted, mapsto] & b_4 &
\end{tikzcd}$$

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$$
\begin{tikzcd}
 & & & a4 & \\
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5 \\
 & & b3 \arrow[r, dotted] & b4 \arrow[uuu, dotted, bend right] &
\end{tikzcd}
$$

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$$\begin{tikzcd}
 & & & a_4 & \\
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5 \\
 & & b_3 \arrow[r, dotted] & b_4 \arrow[r, dotted] \arrow[uuu, dotted, bend right] & 0
\end{tikzcd}$$

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$$
\begin{tikzcd}
 & & & a_4 \arrow[r, dotted] & 0 \\
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5 \\
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$$\begin{tikzcd}
 & & a3 \arrow[r, dotted] & a4 \arrow[r, dotted] & 0 \\
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
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\end{tikzcd}$$

Suppose that $f_3(a_3) = b'_3$, then $\beta_3(b_3) - \beta_3(b'_3) = b_4 - b_4 = 0$. Assume: $\beta_3(b_3) = 0$ (we just subtract an element with preimage in A_3).

$$\begin{tikzcd}
 & & a3 \arrow[r, dotted] & a4 \arrow[r, dotted] & 0 \\
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
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$$
\begin{tikzcd}
 & & a3 \arrow[r, dotted] & a4 \arrow[r, dotted] & 0 \\
A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
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$$

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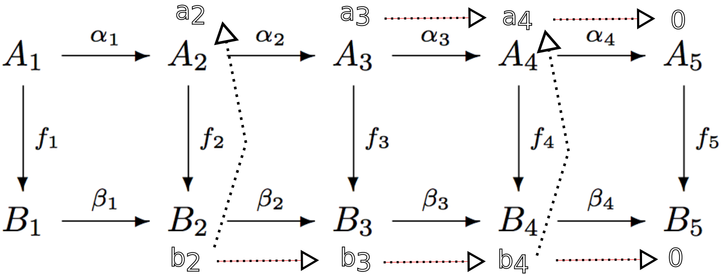

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 & b2 \arrow[r, dotted] & b3 \arrow[r, dotted] & b4 \arrow[r, dotted] & 0
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The Use of Commutative Diagrams

- Commutative diagrams do not present the problems raised by the use of topological diagrams.
- No appeal to spatio-temporal intuition.
- It is easily seen how the diagrams could be eliminated.
  - Our understanding would be compromised!
  - The fact that they are easily coded does not mean that they are dispensable.

As the name suggests, they are still diagrams!

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Individuation of Proofs

Tim Gowers admits that "even the seemingly more basic question, 'When are two proofs the same?' was pretty hard to answer satisfactorily." (2007)

Examples:

1. Euclid's proof that there are infinitely many primes
2. Proofs involving lemmas
3. Minor changes of notations

Evidence for the fact that there are no context-independent identity conditions for proofs. It depends on our interests!

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Sensitivity to Actual Reasoning

If we individuate proofs in a way that is sensitive to the particular reasoning needed to grasp how they support their conclusion, then going from a topological proof involving visualizations to a formal proof in an unique logical language would be a significant change since the properly topological reasoning would be lost.

- This does not mean that the proof cannot be "translated" at all; but that the proof that we would associate would not preserve the topological reasoning.
- Even for knot diagrams, it is easy to see how a purely symbolic proof could be obtained from it.
- Even easier is the case of algebraic diagrams.

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Varieties of Representations in Mathematics

- **1-dimensional**: linear
  1. non-topo-geometric – e.g. algebraic expressions, written natural language.
  2. topo-geometric – e.g. 'linear maps.'

- **2-dimensional**: diagrammatic (not unconstrained 2-dimensional representations, such as illustrations)
  1. non-topo-geometric – e.g. commutative diagrams.
  2. topo-geometric — e.g. knot diagrams.

Main Contributions

- Definition of diagrams as interpreted representations belonging to a specific practice:
  1. 2-dimensionality
  2. constraints on their interpretation and operative dimension

- Explained that even if intuition can lead us astray, it is acceptable in specific mathematical contexts.
  - Enhanced Manipulative Imagination
- Illustrated a variety of diagrams some of which do not exploit intuition at all.

The crisis in intuition should not lead to a ban of diagrams!

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THANK YOU!